Using Katsurada's determination of the Eisenstein series to compute Siegel eigenforms in degree three

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Computational Aspects of L-functions
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Computational Aspects of L-functions

1. Part *I*. What are Siegel modular forms?
2. Part *II*. What are examples of Siegel modular forms?
3. Part *III*. What good are Siegel modular forms?
4. Part *IV*. How are we going to compute Euler factors in degree three?
5. Part *V*. What Euler factors of Siegel modular forms have been seen?

6. You can see some data at: math.lfc.edu/∼yuen/genus3

Siegel Modular Forms

$$R \subseteq \mathbb{R} \text{ is a commutative subring and } J = \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right].$$

- General Symplectic group

$$\mathrm{GSp}_n^+(R) = \{\sigma \in \mathrm{GL}_{2n}(R) : \exists \nu \in R^+ : \sigma' J \sigma = \nu J\}$$

- Similitude: $\nu : \mathrm{GSp}_n^+(R) \to R^+$ given by $\sigma \mapsto \sqrt[n]{\det(\sigma)}$
- Symplectic group

$$\mathrm{Sp}_n(R) = \ker(\nu) = \{\sigma \in \mathrm{SL}_{2n}(R) : \sigma' J \sigma = J\}$$

- Siegel Upper Half Space

$$\mathcal{H}_n = \{\Omega \in M_{n\times n}^{\mathrm{sym}}(\mathbb{C}) : \operatorname{Im}\Omega > 0\}$$

Siegel Modular Forms

- Action of $\sigma = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] \in \mathrm{GSp}_n^+(R)$ on $\Omega \in \mathcal{H}_n$

$$\sigma \cdot \Omega = (A\Omega + B)(C\Omega + D)^{-1}$$

- Factor of Automorphy

$$j : \mathrm{GSp}_n^+(R) \times \mathcal{H}_n \to \mathbb{C}^\times \text{ given by } j(\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right], \Omega) = \det(C\Omega + D)$$

- Cocycle condition: $j(\sigma_1\sigma_2, \Omega) = j(\sigma_1, \sigma_2 \cdot \Omega)\, j(\sigma_2, \Omega)$
- Slash action of group on functions $f : \mathcal{H}_n \to \mathbb{C}$

$$\left(f|_k\sigma\right)(\Omega) = \nu(\sigma)^{nk-n(n+1)/2} j(\sigma, \Omega)^{-k} f\left(\sigma \cdot \Omega\right)$$

Siegel Modular Forms

- Siegel modular group $\Gamma_n = \mathrm{Sp}_n(\mathbb{Z})$
- Vector space of Siegel modular forms of weight k and level one.

$$M_k(\Gamma_n) = \{\text{holomorphic } f : \mathcal{H}_n \to \mathbb{C} : \forall \gamma \in \Gamma_n,\ f|_k\gamma = f,\\ \text{and } \forall Y_o > 0,\ f \text{ is bounded on } \{\Omega : \operatorname{Im}\Omega > Y_o\}\}$$

- Siegel Phi map $\Phi : M_k(\Gamma_n) \to M_k(\Gamma_{n-1})$ given by

$$(\Phi f)(\Omega) = \lim_{\lambda \to +\infty} f\left[\begin{smallmatrix} \Omega & 0 \\ 0 & i\lambda \end{smallmatrix}\right]$$

- Siegel modular cusp forms

$$S_k(\Gamma_n) = \ker(\Phi) = \{f \in M_k(\Gamma_n) : \Phi f = 0\}$$

Fourier expansion

Every Siegel modular form $f \in M_k(\Gamma_n)$ has a Fourier expansion

$$f(\Omega) = \sum_{T:\, T \geq 0,\ 2T \text{ even}} a(T; f) e\left(\langle \Omega, T \rangle\right)$$

- Here, $e(z) = e^{2\pi i z}$ and $\langle \Omega, T \rangle = \mathrm{tr}(\Omega T)$.
- $a(T; \Phi f) = a\left(\left[\begin{smallmatrix} T & 0 \\ 0 & 0 \end{smallmatrix}\right]; f\right)$

Every Siegel modular cusp form $f \in S_k(\Gamma_n)$ has a Fourier expansion

$$f(\Omega) = \sum_{T:\, T > 0,\ 2T \text{ even}} a(T; f) e\left(\langle \Omega, T \rangle\right)$$

Ways to make Siegel Modular Forms

- Eisenstein series
- Theta series
- Polynomials in the thetanullwerte

- Various lifts
- Specializations of symplectic embeddings.
- Generating functions, multiplication, differential operators,...

Siegel Eisenstein Series

$$E_k^{(n)} = \sum_{P_{n,0}(\mathbb{Z})\sigma \in P_{n,0}(\mathbb{Z})\backslash\Gamma_n} 1|_k\sigma \in M_k(\Gamma_n) \quad \text{for } k > n+1.$$

- $P_{n,0}(\mathbb{Z}) = \{\left[\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right] \in \Gamma_n\}$
- $\Phi\left(E_k^{(n)}\right) = E_k^{(n-1)}; \quad \Phi\left(E_k^{(1)}\right) = 1$
- $k > n+1$ ensures absolute convergence on compact sets
- Remarkably, the Fourier coefficients of an Eisenstein series, $a\left(T; E_k^{(n)}\right)$, depend only upon the *genus* of the index T.

The algorithmic computation of the Fourier coefficients of Siegel Eisenstein series $a\left(T; E_k^{(n)}\right)$ began with C. Siegel in the 1930s and was completed by Hidenori Katsurada in 1999.

Example in degree $n = 2$

Weight 4 Eisenstein series of degree 2: $z \in \mathcal{H}_2$

$$
\begin{aligned}
E_4^{(2)}(z) &= 1 + 240\,\mathrm{e}(z_{22}) + 2160\,\mathrm{e}(2z_{22}) + 6720\,\mathrm{e}(3z_{22}) + 17520\,\mathrm{e}(4z_{22}) + \cdots \\
&+ 13440\,\mathrm{e}\langle \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}, z\rangle + 30240\,\mathrm{e}\langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), z\rangle + 138240\,\mathrm{e}\langle \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 2 \end{pmatrix}, z\rangle \\
&+ 181440\,\mathrm{e}\langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right), z\rangle + 604800\,\mathrm{e}\langle \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right), z\rangle + 362880\,\mathrm{e}\langle \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 3 \end{pmatrix}, z\rangle \\
&+ 967680\,\mathrm{e}\langle \begin{pmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & 2 \end{pmatrix}, z\rangle + 497280\,\mathrm{e}\langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & 3 \end{smallmatrix}\right), z\rangle + 1239840\,\mathrm{e}\langle \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right), z\rangle \\
&+ 1814400\,\mathrm{e}\langle \left(\begin{smallmatrix} 2 & 1 \\ 1 & 3 \end{smallmatrix}\right), z\rangle + \cdots
\end{aligned}
$$

omitting $\mathrm{GL}_2(\mathbb{Z})$-equivalent terms

Type II lattices

Unimodular self-dual even lattices

Definition

A lattice Λ in a euclidean space V is **Type II** *means*

- Λ *is even: For all* $u \in \Lambda, \langle u, u \rangle \in 2\mathbb{Z}$.
- Λ *is self-dual:* $\Lambda = \Lambda^* = \{u \in V : \forall v \in \Lambda, \langle u, v \rangle \in \mathbb{Z}\}$.

For a fixed rank, necessarily a multiple of 8, the even unimodular lattices form a single genus.

- rank $= 8$, genus $= \{E_8\}$.
- rank $= 16$, genus $= \{E_8 \oplus E_8, D_{16}^+\}$. (Witt)
- rank $= 24$, genus $= \{$24 Niemeier lattices$\}$. (Niemeier)
- rank $= 32$, $|\text{genus}| > 80$ million.

An infinite family of Type II lattices

The checkerboard lattice: $D_n = \{v \in \mathbb{Z}^n : \sum_{j=1}^{n} v_i \equiv 0 \mod 2\}$.

D_n is even but $[D_n^* : D_n] = 4$.

The glue vector: $[1] = (\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}) \in \mathbb{Q}^n$.

The Type II lattice: $D_n^+ = D_n \cup ([1] + D_n)$.

(In fact, $D_8^+ = E_8$.)

Theta series of Type II lattices

Theorem

Let Λ be a Type II lattice of rank $2k$. The degree n **theta series** *of Λ*

$$\vartheta_\Lambda^{(n)}(\Omega) = \sum_{L \in \Lambda^n} e\left(\frac{1}{2}\langle LL', \Omega\rangle\right) \in M_k\left(\Gamma_n\right)$$

is a Siegel modular form of weight k and degree n.

- $\Phi\left(\vartheta_\Lambda^{(n)}\right) = \vartheta_\Lambda^{(n-1)}$
- $\vartheta_{E_8}^{(n)} = E_4^{(n)} \in M_4(\Gamma_n) = \mathbb{C}\vartheta_{E_8}^{(n)}$, (Duke and Imamoḡlu)
- $\vartheta_{E_8 \oplus E_8}^{(n)} = \vartheta_{D_{16}^+}^{(n)}$ if and only if $n \leq 3$, (Problem of Witt)
- $J_8^{(4)} = \vartheta_{E_8 \oplus E_8}^{(4)} - \vartheta_{D_{16}^+}^{(4)} \in S_8(\Gamma_4)$ is the 1888 Schottky form. (Igusa)
- The Wall: $\sum_{n=1}^{32} \dim S_{16}(\Gamma_n) > 80,000,000$.

Fourier coefficients of Theta series

Generating functions for lattice counts are Siegel modular forms

$$\frac{a\,(T;\vartheta_\Lambda)}{|\operatorname{Aut}_{\mathbb{Z}}(T)|} = \text{Number of sublattices } \tilde{\Lambda} \subseteq \Lambda \text{ with Gram matrix } 2T.$$

Example: $a\left(\frac{1}{2}\left[\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right]; \vartheta_{E_8}\right) = 13\,440 = 12 \cdot 1120 = |\operatorname{Aut}_{\mathbb{Z}}\left[\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right]| 1120$

There are 1120 sublattices $\tilde{\Lambda} \subseteq E_8$ with a basis (v_1, v_2) that satisfies $\begin{pmatrix} \langle v_1, v_1 \rangle & \langle v_1, v_1 \rangle \\ \langle v_2, v_1 \rangle & \langle v_2, v_2 \rangle \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

There are 1120 sublattices of type A_2 inside E_8.

This motivates the whole theory of Siegel modular forms.

Siegel's Theorem

Theorem (Siegel)

Let k be divisible by 4 and satisfy $k > n+1$. We have

$$\left(\sum_{[\Lambda]} \frac{1}{|\operatorname{Aut}_{\mathbb{Z}} \Lambda|}\right) E_k^{(n)} = \sum_{[\Lambda]} \frac{1}{|\operatorname{Aut}_{\mathbb{Z}} \Lambda|} \vartheta_\Lambda^{(n)},$$

where the sum is over isomorphism classes of Type II latices.

1. We can get a similar theorem for $k \equiv 2 \mod 4$ by attaching pluriharmonic polynomials Q to the theta series $\vartheta_{\Lambda,Q}^{(n)}(\Omega) = \sum_{L \in \Lambda^n} Q(L) e\left(\frac{1}{2}\langle LL', \Omega\rangle\right)$ but let's skip the details.
2. The Eisenstein series is naturally associated to the Type II genus.

What good are Siegel Modular Forms?

Uses in Algebraic Geometry

$$\text{Satake Compactification: } \mathcal{S}\left(\Gamma_n \backslash \mathcal{H}_n\right) = \operatorname{proj}\left(\oplus_{k=0}^{\infty} M_k(\Gamma_n)\right)$$

$$\text{Smooth Compactification: } \hat{\mathcal{S}}\left(\Gamma_n \backslash \mathcal{H}_n\right) = \operatorname{proj}\left(\text{valuation subring}\right)$$

The Schottky form $J_8^{(4)} = \vartheta_{E_8 \oplus E_8}^{(4)} - \vartheta_{D_{16}^+}^{(4)} \in S_8(\Gamma_4)$ has the Jacobian locus as its zero divisor in degree $n = 4$.

$\mathbb{C}^4/(\Omega\mathbb{Z}^4 + \mathbb{Z}^4)$ is the limit of Jacobians of compact Riemann surfaces of genus 4 if and only if $J_8^{(4)}(\Omega) = 0$

What good are Siegel Modular Forms?

They make L-functions

- Both $M_k(\Gamma_n)$ and $S_k(\Gamma_n)$ have a basis of Hecke eigenforms.
- But how are we going to compute these spaces in order to make our L-functions?
- The difficulty is in getting enough Fourier coefficients to break the space into eigenspaces and to compute Euler factors.

The Witt Map

A particular symplectic embedding.

The embedding

$$\begin{aligned} W_{ij} : \mathcal{H}_i \times \mathcal{H}_j &\to \mathcal{H}_{i+j} \\ (\Omega_1, \Omega_2) &\mapsto \left[\begin{smallmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{smallmatrix}\right] \end{aligned}$$

pulls back the the **Witt Map**

$$\begin{aligned} W_{ij}^* : M_k(\Gamma_{i+j}) &\to M_k(\Gamma_i) \otimes M_k(\Gamma_j) \\ (\Omega \mapsto f(\Omega)) &\mapsto ((\Omega_1, \Omega_2) \mapsto f\left[\begin{smallmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{smallmatrix}\right]) \end{aligned}$$

The Witt map takes cusp forms to cusp forms.

Properties of the Witt Map

Fourier coefficients of $W_{ij}^* f \in M_k(\Gamma_i) \otimes M_k(\Gamma_j)$ in terms of $f \in M_k(\Gamma_{i+j})$:

$$a\left(T_1 \times T_2; W_{ij}^* f\right) = \sum_{R \in \frac{1}{2} M_{i \times j}(\mathbb{Z})} a\left(\left[\begin{smallmatrix} T_1 & R \\ R' & T_2 \end{smallmatrix}\right]; f\right)$$

1. Pay attention to the fact that R has ij entries. Looping over these entires is the cost of evaluating the Witt map.
2. $W_{ij}^* \vartheta_\Lambda^{(i+j)} = \vartheta_\Lambda^{(i)} \otimes \vartheta_\Lambda^{(j)}$

The theta series have a beautiful decomposition under the Witt map. The decomposition of the Eisenstein series under the Witt map is also beautiful but more subtle.

Garrett's formula (c.1980)

Garrett's Formula

For even $k > 2n+1$

$$W_{nn}^{*}E_k^{(2n)} = \sum_{\ell=1}^{d} c_\ell \, h_\ell \otimes h_\ell$$

$\{h_1, \cdots, h_d\}$ *Hecke eigenform basis of* $M_k(\Gamma_n)$

all c_ℓ *nonzero (special values of L-functions)*

(That is, $E_k^{(2n)}\left(\left(\begin{smallmatrix} z_1 & 0 \\ 0 & z_2 \end{smallmatrix}\right)\right) = \sum c_\ell h_\ell(z_1) h_\ell(z_2)$ for all z_1, z_2)

Degree n: we don't know the dimn d, basis $\{h_\ell\}$, or special values $\{c_\ell\}$...

...but **all** are built into *only-some-of* the **one** Siegel modular form $E_k^{(2n)}$ of degree $2n$

And $E_k^{(2n)}$ is computationally tractable: FCs are genus class functions

Astonishing

Plan of action

Title of talk: Using Katsurada's determination of the Eisenstein series to compute Siegel eigenforms in degree three.

- Use the Witt pieces of Eisenstein series to span $M_k(\Gamma_n)$
- By Garrett's formula this will *always* work when $k > 2n+1$.
- We receive special help in degree three because Tsuyumine has computed $\dim M_k(\Gamma_3)$.
- We need to sum over the **nine** entries of R in $a\left(T_1 \times T_2; W_{ij}^* f\right) = \sum_{R \in \frac{1}{2} M_{i\times j}(\mathbb{Z})} a\left(\begin{pmatrix} T_1 & R \\ R' & T_2 \end{pmatrix}; f\right)$.
- Note we first find the sum on the right as a symbolic sum over genus symbols (in lieu of the more expensive matrix reduction).

We must have a fast way to compute FCs of Eisenstein series and Katsurada has provided it.

Eisenstein series Fourier coefficients

Siegel Eisenstein Series Fourier Coefficient Formula

For even degree $n \in \mathbb{Z}_{>0}$, even weight $k > n+1$, definite T of rank n, write $(-1)^{n/2}\det(2T) = D_T f_T^2$ for a fundamental discriminant D_T and $f_T \in \mathbb{N}$.

$$a\left(T; E_k^{(n)}\right) = \frac{2^{\frac{n}{2}} L\left(1-(k-\frac{n}{2}); \chi_{D_T}\right)}{\zeta(1-k)\prod_{j=1}^{\frac{n}{2}} \zeta(1-(2k-2j))} \prod_{p \mid f_T} F_p\left(T; p^{k-n-1}\right).$$

For odd degree n,

$$a\left(T; E_k^{(n)}\right) = \frac{2^{\frac{n+1}{2}}}{\zeta(1-k)\prod_{j=1}^{\frac{n-1}{2}} \zeta(1-(2k-2j))} \prod_{2p \mid \det(2T)} F_p\left(T; p^{k-n-1}\right).$$

What are these F_p polynomials?

Katsurada's 1999 article

1. *An explicit formula for Siegel Series* (1999) H. Katsurada
   Katsurada writes recursion formulae for the F_p polynomials.
2. *A mass formula for unimodular lattices with no roots* (2003) O. King
   King writes a LISP program to implement Katsurada's recursion formula for the F_p polynomials, and kindly shares it with us.
3. We modify King's program to accept higher degrees.
   (So if it is wrong— we might have done it.)

Definition of F_p polynomials

For a given prime p and half-integral matrix B of degree n over $\mathbb{Z}_p$ define the local Siegel series by

$$b_p(B,s) = \sum_{R \in V_n(\mathbb{Q}_p)/V_n(\mathbb{Z}_p)} e_p(\langle B, R\rangle)\,[R\mathbb{Z}_p^n + \mathbb{Z}_p^n : \mathbb{Z}_p^n]^{-s}$$

$$\xi_p(B) = \chi_p\left((-1)^{n/2}\det(2B)\right).$$

$$\gamma_p(B,X) = \begin{cases} (1-p^{n/2}\xi_p(B)X)^{-1}(1-X)\prod_{i=1}^{n/2}(1-p^{2i}X^2), & n \text{ even},\\ (1-X)\prod_{i=1}^{(n-1)/2}(1-p^{2i}X^2), & \text{if } n \text{ is odd}.\end{cases}$$

Lemma (Kitaoka)

For a nondegenerate half-integral matrix B of degree n over $\mathbb{Z}_p$ there exists a unique polynomial $F_p(B,X)$ in X over $\mathbb{Z}$ with constant term 1 such that

$$b_p(B,s) = \gamma_p(B,p^{-s})F_p(B,p^{-s}).$$

Standard L-function of a Siegel Hecke eigenform

A Hecke eigenform f has L-functions of many sorts

For each p, a *Satake parameter*

$$\alpha_p = (\alpha_{0,p}, \alpha_{1,p}, \cdots, \alpha_{n,p}) \qquad \text{each } \alpha_{i,p} \in \mathbb{C}$$

describes the eigenform behavior of f under $T(p)$ and the $T_i(p^2)$

Euler factor

$$Q(\alpha, X) = (1 - X) \prod_{i=1}^{n} (1 - \alpha_i X)(1 - \alpha_i^{-1} X)$$

Standard L-function

$$\begin{aligned} L^{\mathrm{st}}(f, s) &= \prod_p Q(\alpha_p, p^{-s})^{-1} \\ &= \prod_p \left((1 - p^{-s})^{-1} \prod_{i=1}^{n} (1 - \alpha_{i,p} p^{-s})^{-1} (1 - \alpha_{i,p}^{-1} p^{-s})^{-1} \right) \end{aligned}$$

What Euler factors did people find?

Degree Two

- Kurokawa discovers lifts. (1978)
- $L(F, s, \mathrm{spin}) = \zeta(s-k+1)\zeta(s-k+2)L(f, s)$
  Here eigenform $f \in S_{2k-2}(\Gamma_1)$ and $F \in S_k(\Gamma_2)$.
- Kurokawa and Skoruppa have to compute up to $S_{20}(\Gamma_2)$ to find a nonlift!

- Congruences between lifts and nonlifts also found.

What Euler factors did people find?

Degree Three

- Miyawaki discovers lifts in degree three. (1992)

Miyawaki lifts

Standard L-function Euler factors sometimes decompose recognizably, showing that a Siegel modular form arises from smaller ones. Especially in degree $n = 3$:

Miyawaki Conjectures

1. *For k even, for cusp eigenforms $f \in \mathcal{S}_{2k-4}(\Gamma_1)$ and $g \in \mathcal{S}_k(\Gamma_1)$, there exists a cusp eigenform $h \in \mathcal{S}_k(\Gamma_3)$ such that*

$$L^{\mathrm{st}}(h, s) = L(f, s+k-2)L(f, s+k-3)L^{\mathrm{st}}(g, s)$$

2. *For k even, for cusp eigenforms $f \in \mathcal{S}_{2k-2}(\Gamma_1)$ and $g \in \mathcal{S}_{k-2}(\Gamma_1)$, there exists a cusp eigenform $h \in \mathcal{S}_k(\Gamma_3)$ such that*

$$L^{\mathrm{st}}(h, s) = L(f, s+k-1)L(f, s+k-2)L^{\mathrm{st}}(g, s)$$

(1) proved by Ikeda, (2) still open

What Euler factors did people find?

Degree Three

- Miyawaki discovers lifts in degree three. (1992)
- $S_{12}(\Gamma_3)$ is one dimensional and an Ikeda-Miyawaki lift.
- $S_{14}(\Gamma_3)$ is one dimensional and a Miyawaki lift of type 2.
- We use Katsurada's determination of the Fourier coefficients of Eisenstein series to compute the Witt images of degree six Eisenstein series (2010-present)
- *Congruences to IkedaMiyawaki lifts and triple L-values of elliptic modular forms* (2014) (Ibukiyama, Katsurada, PY).

What Euler factors did people find?

Degree Three

- $\dim S_{16}(\Gamma_3) = 3$. Two conjugate Ikeda-Miyawaki lifts, f_1 and f_2 and an f_3 with unimodular Satake parameters.
- $f_1 \equiv f_3$ modulo a prime above 107.

T	f_1	f_3
$T(2)$	$4414176 + 23328\sqrt{18209}$	-115200
$T_0(4)$	$55296(-17632637 + 1160109\sqrt{18209})$	-784548495360
$T_1(4)$	$-4718592(-1757519 + 1503\sqrt{18209})$	-1062815662080
$T_2(4)$	$1207959552(-209 + 9\sqrt{18209})$	-352724189184
$T_3(4)$	68719476736	68719476736

What Euler factors did people find?

Degree Three

- $\dim S_{18}(\Gamma_3) = 4$.
- Two conjugate Ikeda-Miyawaki lifts, and and two conjugate lifts of (apparently) Miyawaki's second type.
- No congruences over big primes.

What Euler factors did people find?

Degree Three

- $\dim S_{20}(\Gamma_3) = 6$.
- Three conjugate Ikeda-Miyawaki lifts, f_1, f_2, f_3, and and two conjugate lifts f_4, f_5, of (apparently) Miyawaki's second type, and one eigenform f_6 with unimodular Satake parameters.
- $f_1 \equiv f_6$ modulo a prime over 157

for $k = 20$, a standard 2-Euler factorof f_6 is

$$\frac{1}{68719476736}(68719476736 + 183681155072x + 257889079808x^2 + \\ 277369629719x^3 + 257889079808x^4 + 18681155072x^5 + 68719476736x^6)$$

What Euler factors did people find?

Degree Three: this one needs verification.

- $\dim S_{22}(\Gamma_3) = 9$.
- Three conjugate Ikeda-Miyawaki lifts, f_1, f_2, f_3, and and three conjugate lifts f_4, f_5, f_6, of (apparently) Miyawaki's second type, and three conjugate eigenforms f_7, f_8, f_9 with unimodular Satake parameters.
- $f_1 \equiv f_7$ modulo primes over 67 and 613
- $f_4 \equiv f_7$ modulo a prime over 1753

Summary of degree $n = 3$

At least up to weight $k \leq 22$.

- It looks like Miyawaki found all the possibilities for lifts in degree three.
- Can anyone prove Miyawaki's second type of lift?
- The congruence primes come from algebraic triple L-values. Thus, although not all eigenforms are lifts, so far are eigenforms are *explained* by lifts.
- The **Kitaoka** space is the subspace of Siegel modular forms whose Fourier coefficients depend only on the genus of the index. Kitaoka proved this subspace is stable under the Hecke algebra.

  So far, the Eisenstein series is the only element of $M_k(\Gamma_3)$, for $k \leq 22$, that is in the Kitaoka space.

This talk was really about computing Witt images from the Kitaoka space.

Euler factors in degree $n = 4$

in weight $k \leq 16$.

The dimensions and eigenforms for $10 \leq k \leq 16$ were proven by P.Y. and the 2-Euler-factors were worked out by Ryan-P.-Y.

- $\dim S_8(\Gamma_4) = 1$, (Salvati-Manni) Ikeda lift.
- $\dim S_{10}(\Gamma_4) = 1$. Ikeda lift.
- $\dim S_{12}(\Gamma_4) = 2$. One Ikeda lift, one Ikeda-Miyawaki lift.)
- $\dim S_{14}(\Gamma_4) = 3$. Two Ikeda lifts, one Ikeda-Miyawaki lift.
- $\dim S_{16}(\Gamma_4) = 7$. Two Ikeda lifts, two Ikeda-Miyawaki lifts and three forms having the following standard 2-Euler factors, on the following slide.

Standard Euler factors in degree $n = 4$

in weight $k = 16$.

$$\begin{aligned}
&2^{-28}(1-x)(32768 - 5280x - 20755x^2 - 2640x^3 + 8192x^4) \\
&\quad (8192 - 2640x - 20755x^2 - 5280x^3 + 32768x^4) \\
&2^{-34}(1-x)(-2048 + (-1035 + 27\sqrt{18209})x - 4096x^2) \\
&\quad (-4096 + (-1035 + 27\sqrt{18209})x - 2048x^2) \\
&\quad (2048 + 36x + 1601x^2 + 36x^3 + 2048x^4)
\end{aligned}$$

and its conjugate

Will we see a nonlift in weight 18 if we embark on computing the eigenforms using the methods in this talk?

Thank you!